The design of slender columns

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A graphical method is proposed which allows rapid and accurate analysis of slender columns under both axial and eccentric loads, for linear and non-linear structural materials. After demonstrating agreement with existing theory for steel stanchions, the method is used to analyse the behaviour of reinforced concrete columns. The results are compared with present design recommendations in CP114 and CP110 and also with test results. Following consideration of safety factors and the effects of eccentric loadings, new design rules for reinforced concrete columns are proposed. These are shown to approximate well to rigorous theory and test results, yet are simple to use. Suggestions are made for possible further work.

Introduction

The theory of buckling of columns has a long history: the buckling load for a perfect pin-ended elastic strut was established by a Swiss mathematician, Euler, in 1759

\[ P = \frac{\pi^2 EI}{L^2} \]  

where \( P \) is the axial load, \( E \) is the elastic modulus, \( I \) is the moment of inertia and \( L \) is the effective length.

2. The situation where the material has a finite yield strength is more complex and has been the subject of considerable research. The development of rational theories for this has been well summarized by Godfrey. The formula for stanchion strength in BS449 is based on the work of the Steel Structures Research Committee in the 1930s, and is now arranged to provide a constant load factor against failure of 1.7, based on elastic theory and on the assumption that initial imperfections amount to a bow of \( e_o/h = 0.6(L/100h)^2 \), where \( e_o \) is the eccentricity and \( h \) is the section depth. However, tidy formulae are not available where behaviour is non-elastic, e.g. in reinforced concrete, or masonry. Design formulae in codes of practice for these materials are generally approximate and empirically based.

3. The method for an exact theoretical solution is known—the relationship between moment and curvature for the section under its axial load is calculated, and this is then compared with the relationship between curvature and buckling deflexion to establish whether equilibrium can be satisfied ('model column method'). In this form it requires a laborious iterative solution to determine the maximum load which can be carried.

4. A graphical method is proposed here which, once section moment-curvature relationships are known, allows the capacity of columns of any combination of slenderness and initial eccentricity to be established simply and directly. This allows the establishment of practical design rules which have a clear theoretical basis for columns of reinforced concrete or any other non-linear material.

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Method for determining strength of slender columns

5. For any section it is possible to calculate the relationship between moment and curvature under various axial loads: assuming plane sections remain plane, the distribution of stress across the section for a given curvature and neutral axis position can be determined; integration of stresses then gives the corresponding axial force and moment. If this is repeated for a variety of curvatures and neutral axis positions, the relationship between curvature \( \frac{1}{R} \) and moment \( M \) may be established for various values of direct load \( P \). This relationship can then be plotted for \((0.1, 0.2, 0.3, \ldots, 1.0)\) times the maximum axial load \( P_0 \), in the conventional fashion.\(^4\)

6. If we assume that the curvature of a pin-ended column follows a sine curve as it buckles, then the mid-height displacement and curvature are related geometrically by

\[
e_2/h = \frac{1}{\pi^2} \frac{h}{R} \left( \frac{L}{h} \right)^2
\]

where \( e_2 \) is the displacement, \( L \) is the length and \( R \) is the radius of curvature. This relationship can be plotted for various column lengths (Fig. 1).

7. By comparing the moment \( P \cdot e_2 \) with the moment resistance at various values of curvature and axial load, the maximum load for which equilibrium is possible can be found by trial and error. However, this is time-consuming, particularly if a general investigation of slender column behaviour is to be attempted. A direct solution (without iteration) would be highly preferable. This can be achieved by a simple rearrangement of the information shown on the graphs. Instead of plotting curvature against moment, it is plotted against load eccentricity. Fig. 2

![Graph showing relationship between buckling deflexion and curvature](image)

*Fig. 1. Relationship between buckling deflexion and curvature*
Fig. 2. Relationship between load eccentricity and curvature, 40 N/mm² column, +0.8%, long-term load

shows such a graph, prepared for a 40 N/mm² column with 0.8% high-yield steel, under long-term load (steel at 0.2 h and 0.8 h, stress–strain curves as in Figs 3(a) and 3(b), with long-term concrete strains taken as 2 1/2 times short-term values.)

8. At equilibrium, the load eccentricity must equal the column's displacement; if a line relating curvature to displacement is superimposed on Fig. 2, the maximum load for which equilibrium may be attained can be read off directly (Fig. 4). If an initial eccentricity $e_1$ is present this can be incorporated by displacing A to B, with failure occurring at total eccentricity $e = e_1 + e_2$.

9. For greatest convenience, curvature-displacement lines for various slender-ness ratios are plotted on a transparent overlay, incorporating suitable allowances for initial imperfections (Fig. 5). Then, for any section, once the load eccentricity-curvature relationships are established, the maximum load it can sustain for any slenderness ratio can be read off directly by superimposing the overlay (Fig. 6). The effect of various initial load eccentricities can be determined by simply displacing the overlay vertically to the appropriate value of $e/h$.

10. Where moments are applied to a column, these may give an initial deflex-ion somewhat different from that calculated by equation (2), which corresponds with a sine-curve variation along the column's length. If the moment is uniform throughout the column length, the equation becomes $e_1/h = \frac{P}{P_0} \frac{h}{R} (L/h)^2$; if it is triangular, due to a lateral load at mid-height, the equation becomes $e_1/h = \frac{1}{4} (h/R)(L/h)^2$. Where this would make a significant difference, it can be dealt with as shown in Fig. 7, which shows the analysis of a pin-ended column with a uniform initial load eccentricity of 0.1 h throughout its height, for comparison with Fig. 4 (some trial and error is necessary).
Fig. 3. (a) Short-term stress–strain curve: concrete; (b) stress–strain curve: high yield reinforcement

11. For interest, a solution for steel stanchions of grade 43 steel ($Y_s = 255$ N/mm$^2$), based on elastic theory and the BS449 assumed imperfections, is shown in Fig. 8. For comparison, Table 1 gives the results from Table 17(a) of BS449, referred to $Y_s/1.7 = 150$ N/mm$^2$.

12. Although this method of analysis is relatively simple (the major labour is in the preparation of the load eccentricity–curvature graphs), it is accurate and powerful, making it practical to investigate the capacity of pin-ended slender

Fig. 4. Graphical analysis of column ($L/h = 20$): $A: e_1 = 0$; $B: e_1 = 0.1h$
Fig. 5. Overlay showing buckling deflexion against curvature, with allowance for initial imperfections $e_0 = 0.0015L$.

Fig. 6. Fig. 5 superimposed on Fig. 4 to show relationship between load capacity and slenderness of column.
BEAL curve (b) to establish maximum load (cf. $P = 0.4P_0$, Fig. 4).

Fig. 7. Exact solution for column ($L/h = 20$), with constant applied moment throughout its height ($e_1 = 0.1h$)

Fig. 8. Graphical analysis of Grade 43 steel stanchion
columns of non-linear materials under both axial and eccentric loads. Although it was devised for reinforced concrete, it may also be applied to columns of any other materials.

**Slender reinforced concrete columns—CP114 and CP110**

13. The behaviour of reinforced concrete is complex—steel reinforcement with relatively simple elastic properties is combined with concrete, which has a non-linear stress-strain curve, creep and a tendency to crack. Current codes of practice (CP114 and CP110) give different recommendations for slender columns and neither has the kind of rigorous theoretical basis of BS449's recommendations for steel stanchions. CP114 recommends a set of capacity reduction factors related to slenderness (Table 18 of the code). While these are simple to use, their derivation is obscure—for L/h values up to 33, the values are the same as those proposed by the Reinforced Concrete Structures Committee (RCSC) in 1933. The values seem to have survived unchanged despite changes in concrete, reinforcement and design stresses, and despite criticism of the RCSC proposals by Thomas as long ago as 1939.

14. CP110 (Clause 3.5.7) takes a different approach. Rather than reducing column capacity with slenderness, it defines an additional moment to be applied in ultimate section design to cover the effects of buckling. The recommendations are based on theoretical work and comparison with experimental results and their development has been described by Cranston. For interest, axially loaded column capacities according to the various recommendations are compared in Fig. 9 for a

![Diagram](image-url)

*Fig. 9. Relationship between axial load capacity and slenderness—comparison of methods (40 N/mm² reinforced concrete column)*
BEAL

40 N/mm² rectangular column with 1% or 4% high-yield steel in four bars placed at \( d = 0.2h \) and \( 0.8h \).

15. Table 18 of CP114 is open to criticism for its vintage and lack of defined theoretical basis. However, CP110's recommendations are also open to criticism: calculations based on the ultimate moment resistance of a cracked section cannot be directly related to the instability failure of a slender column which, as is clear from equation (1), depends on stiffness rather than strength. In particular, the CP110 recommendations predict that an increase in reinforcement will lead to a disproportionate increase in carrying capacity for slender columns, and that buckling effects are minimal until \( L/h \) values of well over 20 are attained. Both of these seem dubious when basic buckling considerations and the behaviour of other elements, such as steel stanchions, are considered.

16. In Cranston's report, the CP110 method is compared with a large number of experimental results. The ratios of test load/predicted load for short-term tests range from 0.60 to 7.00 and have a mean of 1.46 and a standard deviation of 0.80, giving a coefficient of variation of 0.55; for long-term tests, the results range from 0.56 to 1.42, giving a mean of 0.94 and a coefficient of variation of 0.22. In the report, a comparison with CP114's method is not given but calculations show that the range, mean and coefficient of variation of the corresponding ratios are 0.45–4.83, 1.23 and 0.47 (short-term tests) and 0.41–1.18, 0.71 and 0.28 (long-term tests). Although some of the observed variations can be attributed to uncertainties over test details, etc., it seems that the agreement between theory and experiment is not particularly good, and that claims for a clear theoretical superiority of CP110 over CP114 are open to doubt.

Analysis of slender reinforced concrete columns

17. For this Paper, the capacities of rectangular columns with reinforcement at \( 0.2h \) and \( 0.8h \) have been calculated by the method set out earlier. This has been done for 20 N/mm² and 40 N/mm² concrete, each with either 0.8% or 4% of high-yield (425 N/mm²) longitudinal reinforcement, and the calculations have been done for both short- and long-term conditions. For the concrete, the parabolic short-term stress–strain curve proposed in CP110 has been used (Fig. 3(a)); this gives figures for the elastic modulus which are in reasonable agreement with those quoted by Faber and Appendix A of CP110. For the steel reinforcement, the approximate stress–strain curves proposed in CP110 have been used (Fig. 3(b)).

18. While short-term column capacity is easiest to calculate and to compare with test results, most of the load on concrete columns in real structures is usually permanent and this long-term capacity is more relevant to design. Creep, which effectively reduces the stiffness of concrete members subjected to long-term loads, is very variable, depending on age at loading, duration of load and many other factors. The variety of available recommendations is quite wide and there is also the question of what concrete age and load duration it would be reasonable to assume—according to figures in the CP110 handbook, the effective elastic modulus varies between 27% (28 day-old column subjected to permanent load) and 51% (100 day-old column loaded for 100 days) of the short-term values. The 'best' value to take is a matter of judgement; on the basis that full load is not usually applied till some time after construction, and that an overload is not likely to last for the entire life of the building, a value of 40% of the short-term modulus
DESIGN OF SLENDER COLUMNS

Table 2. Ratio $P/P_0$ for columns with high-yield steel at $d = 0.8h$, $e_0 = 0.0015L$

<table>
<thead>
<tr>
<th>Concrete grade, N/mm²</th>
<th>Reinforcement</th>
<th>$L/h$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>10</td>
</tr>
<tr>
<td>Short term</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>0.8%</td>
<td>0.89</td>
</tr>
<tr>
<td>40</td>
<td>4.0%</td>
<td>0.89</td>
</tr>
<tr>
<td>Long term</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>0.8%</td>
<td>0.90</td>
</tr>
<tr>
<td>40</td>
<td>4.0%</td>
<td>0.91</td>
</tr>
</tbody>
</table>

has been used here (this approximately corresponds to the maximum load being applied for almost three years to a column which is at least 100 days old).

19. The question of initial eccentricity due to the column being bowed or out of plumb is similarly difficult: not only are there concrete tolerances but the reinforcement can also be misplaced. Published data on this appear to be scant—enquiries by the Author failed to locate any in the UK, for example. Redekop\(^\text{12}\) studied the question in Canada but did not arrive at a relationship between column bow and length. In the absence of better information, a value of $e_0 = 0.0015L$ is suggested here. This corresponds to a bow of 4.5 mm on a 3 m braced column, or 9 mm out of plumb on an unbraced one. Conservative assumptions about the position of reinforcement in columns are advisable, hence the adoption of $d = 0.8h$. The graphs prepared for the four different sections under short- and long-term loading are given in Figs 10 and 11.

20. For practical reasons, the section analyses were done by dividing the section at intervals of 0.2h, determining the stresses at these points and integrating these stresses (by trapezoidal rule) to calculate the force and moment corresponding to a given curvature. Points in between the calculated values were obtained by interpolation. Thus it should be noted that the curves are to some extent approximate, and that a more precise analysis might alter the results slightly.

21. The calculated reductions in column capacities resulting from the effects of slenderness are given in Table 2.

Reinforced concrete columns with bending

22. Both CP114 and CP110 assume that the allowance for slenderness does not need to be altered when the applied load is eccentric. However, calculations suggest that this is not so: as illustrated in Table 3, slenderness affects these columns more seriously than axially loaded ones.

23. At first sight, these figures are worrying, suggesting that present methods might be unsafe; however, the real circumstances which give rise to column moments must be considered.

Braced frame, where beam bending causes column moments

24. In this situation, maximum moments usually occur at the column ends, with much lower moments at mid-height; this gives a hidden safety margin when design is based on moments at the ends (or at one-eighth of the height from the
Fig. 10. (Above and below) Analysis of short-term capacity of reinforced concrete columns: (a) 20 N/mm$^2$ + 0.8%; (b) 20 N/mm$^2$ + 4%; (c) 40 N/mm$^2$ + 0.8%; (d) 40 N/mm$^2$ + 4%
Fig. 11. (Above and below) Analysis of long-term capacity of reinforced concrete columns: (a) 20 N/mm² + 0.8%; (b) 20 N/mm² + 4%; (c) 40 N/mm² + 0.8%; (d) 40 N/mm² + 4%
Table 3. Ratio $P/P_e$ for load applied at eccentricity $e_1 = 0.3h$ ($P_e$ is short column strength for $e_1 = 0.3h$)

<table>
<thead>
<tr>
<th>Concrete grade, N/mm²</th>
<th>Reinforcement</th>
<th>$L/h$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>10</td>
</tr>
<tr>
<td>Long term</td>
<td>20+</td>
<td>0.8%</td>
</tr>
<tr>
<td></td>
<td>20+</td>
<td>4.0%</td>
</tr>
<tr>
<td></td>
<td>40+</td>
<td>0.8%</td>
</tr>
<tr>
<td></td>
<td>40+</td>
<td>4.0%</td>
</tr>
</tbody>
</table>

ends). Even where end moments are of opposite signs, giving constant moment throughout the column height, any tendency for the column to buckle will lead to a reduction or even reversal of moment at the column ends; even without buckling, the effect of slenderness is to reduce the column's bending stiffness, thus reducing the moment it attracts from the beams compared with the theoretical value. In this situation there seems to be little cause for concern.

Unbraced frame subjected to temporary sway loads

25. Sway loads are normally short term—commonly wind load—and thus 'short-term' rather than 'long-term' factors apply. Comparisons suggest that the straightforward use of the long-term coefficients from Table 2 for both axial and eccentric loads gives satisfactory results.

Eccentrically loaded cantilever column or sway frame with long-term lateral load

26. None of the relieving factors listed in §§ 21 and 22 applies here, so there must be some concern that CP114 and CP110 could be unsafe in these situations.

Comparison with tests

27. Reinforced concrete column capacities calculated by the method outlined have not been verified by a programme of tests. However, comparison of the test results listed by Cranston (biaxial bending and those outside the range of the prepared graphs were omitted) with the theoretical capacities (based on $d = 0.8h$, $f_c = 425$ N/mm², without correction for differences in effective depth or steel strength) gave a range, mean and coefficient of variation of 0.78–3.8, 1.41 and 0.31 (short term) and 0.87–1.93, 1.35 and 0.19 (long term); these were an encouraging improvement over the corresponding figures for CP110 and CP114 quoted in § 16. It is likely that a full analysis of the results, based on true steel strengths and locations, would give even better agreement. (For the results in question, CP110 gives means and coefficients of variations of 1.48 and 0.58 (short term) and 0.94 and 0.22 (long term).)

Approximate method for design

28. Although the determination of column strength from load eccentricity-curvature graphs is quick and easy, the preparation of the graphs is not, and simplified methods are necessary for normal design. If we compare the factors in Table 2 with the results of the CP110 additional moment method, it can be seen that the CP110 method greatly exaggerates the effect of reinforcement on slender
DESIGN OF SLENDER COLUMNS

Table 4. Ratio $P/P_e$ for load applied at $e_1 = 0.3h$, approximate method

<table>
<thead>
<tr>
<th>$L/h$</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>35</th>
<th>40</th>
<th>50</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td>20N + 0.8%</td>
<td>0.64</td>
<td>0.48</td>
<td>0.40</td>
<td>0.31</td>
<td>0.24</td>
<td>0.18</td>
<td>0.14</td>
<td>0.09</td>
<td>0.06</td>
</tr>
<tr>
<td>20N + 4.0%</td>
<td>0.76</td>
<td>0.65</td>
<td>0.54</td>
<td>0.42</td>
<td>0.34</td>
<td>0.24</td>
<td>0.18</td>
<td>0.11</td>
<td>0.08</td>
</tr>
<tr>
<td>40N + 0.8%</td>
<td>0.53</td>
<td>0.35</td>
<td>0.28</td>
<td>0.21</td>
<td>0.16</td>
<td>0.12</td>
<td>0.08</td>
<td>0.05</td>
<td>0.04</td>
</tr>
<tr>
<td>40N + 4.0%</td>
<td>0.60</td>
<td>0.54</td>
<td>0.44</td>
<td>0.34</td>
<td>0.26</td>
<td>0.19</td>
<td>0.14</td>
<td>0.09</td>
<td>0.06</td>
</tr>
</tbody>
</table>

column capacity. CP114's reduction-factor presentation, which gives constant factors for all reinforcement proportions, seems a better match and, if the values are varied with concrete strength, a simple table of reduction factors would give a very reasonable approximation to Table 2.

29. Where moments are present the situation is more awkward but, as already established, in most practical situations the application of the basic reduction factor to both axial load and moment should give reasonable results, and in braced columns the CP114 suggestion that sections less than one-eighth of the length from the ends need not have the factor applied also seems reasonable. However, where long-term moments are present in the region of maximum buckling moments and would not be relieved by buckling (§ 23), the analysis suggests that the simple approach might be unsafe. In some cases a full analysis would be best, but often a simple approximate solution is all that is wanted.

30. Trial calculations suggest that reasonable answers can be obtained in this situation if the eccentricity of the applied load is increased by $e_{add}$, where

$$e_{add} = 0 \quad \text{at} \quad L/h = 0$$

$$e_{add} = \frac{P_{cc}}{P_0} \cdot e_1 \geq 0.25h \quad \text{at} \quad L/h \geq 15;$$

$P_0$ is the axial load capacity of the section and $P_{cc}$ is the axial load capacity of the concrete acting alone. For values of $L/h$ between 0 and 15, values may be interpolated. Values of $P/P_e$ for eccentricity $e_1 = 0.3$ recalculated in this manner are given in Table 4 and it can be seen that reasonable agreement with Table 3 has been obtained.

31. The long-term test results quoted in Cranston's report were compared with analysis based on Table 2 and the eccentricity modification given above. This showed a range of results from 0.97 to 1.97 with a mean of 1.28 and a coefficient of variation of 0.16, an encouraging result showing that the proposed design method is considerably more accurate than either CP110 or CP114.

Reinforced concrete columns: safety factors

32. Although the principles for practical design of slender reinforced concrete columns have been established, the values of factors to be used depend on the way safety factors are applied. This is a highly controversial matter at present, with two contending approaches—the stress-factor approach of CP114 (based on working loads) and the partial-factor approach of CP110 (based on factored loads). Both are based on similar plastic section analysis, but they determine and apply the necessary factor of safety in different ways.
33. CP114 requires a safety factor of 1.8 against failure, with concrete in the structure assumed to have 75% of the strength of test cubes. This reflects the less favourable placing, compacting and curing conditions for the concrete in a structure. A value of 75% appears to be a reasonable estimate of the strength difference between air-cured structural concrete and water-cured test cubes and thus, while it might at first appear to be an additional factor of safety, this is not really so.

34. CP110 divides the factor of safety into two partial factors, one applied to loads and the other to materials. The overall load factor depends on the ratio of dead to live load, varying between 1.4 and 1.6; as concrete columns usually carry mainly dead load, their overall load factor is typically about 1.45. The materials factor for concrete is 1.5 but no separate allowance is made for the reduced strength of concrete in the structure noted above; if this is taken as 75%, then the effective safety factor on concrete is similar to that on steel, 1.15. This implies that the overall safety factor is usually about 1.65.

35. When applied to slender columns, further differences emerge: the capacity of a slender column is proportional to stiffness, rather than strength. There is no difficulty with CP114—its safety factor of 1.8 applies to both stocky and slender columns. However, in CP110 the materials factor on concrete stiffness is not 1.5 but \( \sqrt{1.15} \), which is 1.07, and that on steel stiffness is 1.0; the effect in slender columns is to reduce the usual safety factor of about 1.65 to 1.55 where reinforcement is light and to 1.50 where it is heavy. It is most unusual to have a reduced safety factor for slender members—traditionally, the safety factor for these has been the same as or greater than that for stocky ones. It is not certain whether this was the intention of those who devised the partial factor format used in CP110: these very low factors are outside the range (1.61–1.84) quoted in the introduction to CP110 and would not be acceptable to many engineers.

36. For this reason, design proposals derived in accordance with CP110's principles will not be presented here. Instead, the principles of CP114 have been followed in deriving design rules, as there seems little doubt regarding their acceptability. However, there should be no objection to using the resulting rules in CP110 designs.

Proposed design rules for reinforced concrete columns

37. These are based on the following assumptions:

(a) a constant safety factor is used for all slenderness ratios
(b) concrete in a real structure has 75% of the specified cube strength
(c) the concrete has a parabolic stress–strain curves as in Fig. 3(a), with strains taken as \( 2\frac{1}{4} \) times short-term values to allow for creep
(d) the reinforcement is high yield, with a stress–strain curve as in Fig. 3(b)
(e) the column follows a sinusoidal curve as it buckles
(f) initial imperfections are equivalent to a bow of 0.0015L.

38. Although coefficients have been calculated for \( L/h \) up to 60 (slenderness ratio 207), columns as slender as this are very vulnerable to differential temperature, shrinkage, construction tolerances and accidental damage. Because of this, the coefficients for normal use have been tapered out to zero at an \( L/h \) value of 60. The concrete strengths of 20 N/mm\(^2\) and 40 N/mm\(^2\) in the initial analysis correspond to design cube strengths of 27 N/mm\(^2\) and 53 N/mm\(^2\) respectively (see (b) of § 37). Reduction factors based on these have been calculated for design cube
Table 5. Proposed capacity reduction factors for design of slender reinforced concrete columns

<table>
<thead>
<tr>
<th>$L/h$</th>
<th>$L/r_{ty}$</th>
<th>25 N/mm²</th>
<th>50 N/mm²</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>17</td>
<td>0.95</td>
<td>0.95</td>
</tr>
<tr>
<td>10</td>
<td>34</td>
<td>0.88</td>
<td>0.85</td>
</tr>
<tr>
<td>15</td>
<td>52</td>
<td>0.80</td>
<td>0.71</td>
</tr>
<tr>
<td>20</td>
<td>69</td>
<td>0.66</td>
<td>0.56</td>
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<td>0.23</td>
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<td>40</td>
<td>138</td>
<td>0.23</td>
<td>0.18</td>
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<tr>
<td>50</td>
<td>172</td>
<td>0.12</td>
<td>0.09</td>
</tr>
<tr>
<td>60</td>
<td>207</td>
<td>0.0</td>
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</tr>
</tbody>
</table>

strengths of 25 N/mm² and 50 N/mm² and these are given in Table 5. For interest, the values for 40 N/mm² columns are plotted against those from CP110 and CP114 on Fig. 9.

39. These coefficients may also be used for eccentrically loaded columns in the normal fashion, except in the case of unbraced columns where the maximum moments generated by sustained loads occur in the same regions as those due to buckling. Examples of the latter are cantilever columns subjected to sustained moments or sway frames subjected to sustained lateral loads.

40. In these situations, either a rigorous analysis should be adopted or the load eccentricity should be increased by $e_{\text{add}}$, where

$$e_{\text{add}} = \frac{P_{\text{cc}}}{P_0} \cdot e + 0.25h \quad \text{for} \quad L/h \geq 15$$

$$e_{\text{add}} = 0 \quad \text{for} \quad L/h = 0$$

$P_{\text{cc}}$ and $P_0$ are the short column axial load capacity of the concrete alone and of the reinforced section respectively. Intermediate values may be interpolated.

Conclusions

41. A graphical method has been outlined which gives quick and accurate solutions for the capacities of columns over a wide range of slenderness and load eccentricities. The method allows the capacities of columns of non-linear materials to be determined once moment–curvature relationships have been established.

42. The application of the method to reinforced concrete columns allows design rules to be established which have a clear basis in theory, and the results obtained suggest that the use of simple capacity reduction factors (similar to those in CP114) is the most suitable presentation for normal use. Comparison with test results shows that, although the method is simple, it is considerably more accurate than CP110 or CP114. An additional-moment treatment like that in CP110 is misleading at higher slenderness ratios; CP110’s limit-state method of dividing the safety factor into partial factors on loads and materials also seems unsuited to the analysis of slender columns.
Further work

43. If the method presented here is accepted as useful, it should assist in the theoretical examination of various types of compression members. The following areas of work are suggested:

(a) confirming the properties of reinforced concrete sections other than the simple one analysed here—perhaps publishing moment—curvature graphs for general use
(b) analysis of plain and lightly reinforced concrete sections such as walls
(c) brickwork and blockwork walls.

44. It would be of considerable assistance if more definite information could be obtained on the size of initial imperfections which occur in properly built columns and walls in practice, so that these could be allowed for with more confidence.

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